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LETTER TO THE EDITOR

The importance of being odd

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Abstract

In this Letter I mainly consider a finite *XXZ* spin chain with periodic boundary conditions and an *odd* number of sites. This system is described by the Hamiltonian $H_{xxz} = -\sum_{j=1}^{N}\{\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\,\sigma_j^z\sigma_{j+1}^z\}$. As it turns out, the ground state energy is proportional to the number of sites $E = -3N/2$ for a special value of the asymmetry parameter $\Delta = -1/2$. The trigonometric polynomial $Q(u)$, the zeros of which are parameters of the ground state Bethe eigenvector, is explicitly constructed. This polynomial of degree $n = (N-1)/2$ satisfies the Baxter T–Q equation. Using the second independent solution of this equation that corresponds to the same eigenvalue of the transfer matrix, it is possible to find a derivative of the ground state energy w.r.t. the asymmetry parameter. This derivative is closely connected with the correlation function $\langle\sigma_j^z\sigma_{j+1}^z\rangle = -1/2 + 3/2N^2$. This correlation function is related to the average number of spin strings for the ground state $\langle N_{\mathrm{string}}\rangle = \frac{3}{4}(N - 1/N)$. I would like to stress that all the above simple formulae are *not applicable to* the case of an even number of sites which is usually considered.

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I did not care what it was all about. All I wanted to know was how to live in it.
Ernest Hemingway

About 30 years ago Baxter noticed [1] that in some exceptional cases the ground state energy of the *XYZ* spin chain, which has the Hamiltonian

$$H_{xyz} = -\sum_{j=1}^{N}\{J_x\sigma_j^x\sigma_{j+1}^x + J_y\sigma_j^y\sigma_{j+1}^y + J_z\sigma_j^z\sigma_{j+1}^z\} \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1 \tag{1}$$

has the especially simple value

$$\lim_{N\to\infty} E/N = -(J_x + J_y + J_z) \qquad \text{if } J_x,\ J_y \text{ and } J_z \text{ satisfy } J_xJ_y + J_yJ_z + J_zJ_x = 0$$

in the thermodynamic limit. He later noted [2], that the inversion relation gives a very simple eigenvalue for the eight-vertex model transfer matrix that corresponds to (1). Using standard notation for the Boltzmann weights of the eight-vertex model we can reformulate Baxter's remark as follows. If the weights satisfy the condition

$$(a^2 + ab)(b^2 + ab) = (c^2 + ab)(d^2 + ab) \tag{2}$$

then the transfer matrix has (up to a sign) eigenvalue $T = (a + b)^N$. Consequently the Hamiltonian (1), even for finite chains, has an exact eigenenergy of $E = -N(J_x + J_y + J_z)$. A natural suggestion follows. Suppose that the corresponding eigenvector is the ground state vector, then it is probably possible to obtain interesting information for the finite chains. However, there are some problems. It is evident that Baxter's remark is valid for $N = 1$ when both eigenvalues of the transfer matrix are $T = a + b$, but for $N = 2$, the transfer matrix does not have the eigenvalue $T = (a + b)^2$. For simplicity let us consider the six-vertex model which is the trigonometric limit ($d = 0$) of the eight-vertex model. Hamiltonian (1) is reduced to the *XXZ* Hamiltonian with a special asymmetry parameter $\Delta = -1/2$:

$$H_{xxz} = -\sum_{j=1}^{N} \{\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y - \tfrac{1}{2}\sigma_j^z \sigma_{j+1}^z\} \qquad \vec{\sigma}_{N+1} = \vec{\sigma}_1. \tag{3}$$

Solving Baxter's T–Q equation for $N = 2$ we easily find a trigonometric polynomial $Q(u)$ of degree 2, but the corresponding Bethe vector does not exist.

There are at least *three ways* to fix this problem and to obtain a simple eigenvalue in the transfer matrix spectrum. The first two have to do with modifying system (3).

Firstly one can modify the boundary conditions. In 1987 Alcaraz *et al* [3] considered an open *XXZ* spin chain with a special magnetic field at the boundaries. The authors carried out an intensive investigation of the system. For $\Delta = -1/2$ they found a linear dependence between the ground state energy and the number of sites. The Hamiltonian of this chain can be expressed with the help of elements of the Temperlay–Lieb algebra. For $\Delta = -1/2$ this algebra has the trivial one-dimensional representation. In the circumstances it is possible to explain the simplicity of the ground state energy. At present, due to multiple investigations in the 1990s, (see, e.g., the papers of Hinrichsen *et al* [4] and Martin-Delgado and Sierra [5]) this system is considered trivial. I intend to discuss all these questions in a future publication. In the present work I limit myself to a reference to paper [6], where an exact solution of the Baxter's T–Q equation was found. It is of importance that this solution corresponds to the ground state of the system.

Secondly one can apply an additional horizontal field. This field breaks the spin reversal invariance of the original model. As a result, a lot of new Bethe states emerge. For a special value of the field strength one finds in the spectrum of the transfer matrix the simple value $T = -(a + b)^N$ (for even N only). The associated spin chain Hamiltonian is described by Perk and Schultz [7]. I am indebted to Bazhanov and Baxter who informed me about this possibility. It was investigated in [8, section 6], which is an extended version of [6].

The two above-mentioned methods deal with the trigonometric case only. I chose a third way. *As it turns out, it is enough to consider the usual XYZ (XXZ)-spin chain with periodic boundary conditions, but with an ODD number of sites* $N = 2n + 1$. I have checked that for $N = 3, 5, 7$ the transfer matrix for the eight-vertex model has the largest eigenvalue $T = (a + b)^N$ when the weights satisfy condition (2).

Due to technical difficulties typical for the eight-vertex model, I limit myself here to the trigonometric case ($d = 0$). The existence of the above-mentioned simple eigenvalue for the ground state energy was discovered in the trigonometric case by Alcaraz *et al* [25].

In this case the eight-vertex model reduces to the six-vertex model and formula (2) reduces to $c^2 = a^2 + ab + b^2$. The corresponding Hamiltonian is given by (3). *The simplicity of the ground state in this case allows one to find simple explicit formulae for some of the correlations.*

Let us consider a product of z-axis spin projection operators acting at the neighbouring sites. One of the simplest correlations is the average of this product over the ground state

$$C_1^{\parallel}(\Delta) \equiv \langle \sigma_j^z \sigma_{j+1}^z \rangle. \tag{4}$$

Due to translational invariance, this correlation does not depend on j. Similar correlations related to the x and y axes do not depend on j also. Their values coincide, due to z-axis rotational invariance. In what follows we use the notation

$$C_1^{\perp}(\Delta) \equiv \langle \sigma_j^x \sigma_{j+1}^x \rangle = \langle \sigma_j^y \sigma_{j+1}^y \rangle.$$

The simple formulae for these correlations, valid for $\Delta = -1/2$ and for odd N, are the main result of this Letter.

Firstly, let us formulate the starting point of our calculations. It is well known that the Boltzmann weights of the six-vertex model can be conveniently parametrized by spectral parameter u and crossing-parameter η as

$$a = \sin(u + \eta/2) \qquad b = \sin(u - \eta/2) \qquad \text{and} \qquad c = \sin\eta. \tag{5}$$

The asymmetry parameter is related to the crossing parameter via $\Delta = \cos\eta$. For the special case $u = \eta/2$, the transfer matrix $\hat{T}$ is proportional to a shift in the chain by one site and its eigenvalues for all states with zero momentum are $\sin^N \eta$. It is also known that the Hamiltonian for the *XXZ* spin chain is related to the logarithmic derivative of the transfer matrix w.r.t. the spectral parameter:

$$H_{xxz} = -\sum_{j=1}^{N} \{\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \cos\eta\, \sigma_j^z \sigma_{j+1}^z\} = N\cos\eta - 2\sin\eta \left(\frac{\hat{T}_u'}{\hat{T}}\right)_{u=\eta/2}. \tag{6}$$

We will also use Baxter's T–Q equation

$$T(u)\, Q(u) = \sin^N(u + \eta/2)\, Q(u - \eta) + \sin^N(u - \eta/2)\, Q(u + \eta) \tag{7}$$

where $T(u)$ and $Q(u)$ are eigenvalues of transfer matrix $\hat{T}$ and of the auxiliary matrix $\hat{Q}$, corresponding to a common eigenvector. In the trigonometric case

$$Q(u) = \prod_{j=1}^{m} \sin(u - u_j) \tag{8}$$

where u_j satisfy the Bethe equation.

Firstly, we explicitly find for an odd value of N and for a fixed value of the crossing parameter $\eta = 2\pi/3$ two solutions $Q(u)$ and $P(u)$ of equation (7) corresponding to the hypothetical eigenvalue of the transfer matrix $T(u) = (a + b)^N$. We argue that the Bethe vector, constructed with the help of $Q(u)$, corresponds to the ground state. *Then*, using the result of [8], we formulate the relation between these two solutions and the derivative of the largest eigenvalue of the transfer matrix $T(u)$ w.r.t. η. *Finally*, the knowledge of this derivative lets us find the correlation (4).

Let us momentarily fix the crossing parameter $\eta = 2\pi/3$ ($\Delta = -1/2$) and consider the conjectured value $T = (a + b)^N$. Using parametrization (5), one can write this as $T(u) = \sin^N u$. Baxter's equation (7) for odd N takes the cyclic form

$$f(u) + f\left(u + \frac{2\pi}{3}\right) + f\left(u + \frac{4\pi}{3}\right) = 0 \tag{9}$$

where $f(u) = \sin^{2n+1} u\; Q(u)$. To solve the equation we follow [6]. Using cross invariance of the T–Q equation and the simple structure of the transfer matrix for $u = \eta/2$, one can show that $Q(u)$ is an even function and thus $f(u)$ is an odd trigonometric polynomial of degree $3n+1$, satisfying periodicity $f(u+\pi) = (-1)^{n+1} f(u)$. We can therefore write

$$f(u) = a_1 \sin(3n+1)u + a_2 \sin(3n-1)u + a_3 \sin(3n-3)u \ldots.$$

Equation (9) is satisfied if $a_{3\nu} = 0$. This condition implies

$$f(u) = \sum_{k=0}^{n} \alpha_k \sin(1-3n+6k)u.$$

The polynomial $f(u)$ is divided by $\sin^{2n+1} u$ by definition. This fixes the coefficients α_k. It is clear that the first $2n$ derivatives have to be zeros for $u=0$. The derivatives of even order are trivially zeros while the odd derivatives give

$$\sum_{k=0}^{n} \alpha_k (1-3n+6k)^{2\mu+1} = 0 \qquad \mu = 0, 1, \ldots, n-1.$$

This system is equivalent to the condition that the relation

$$\sum_{k=0}^{n} \alpha_k (1-3n+6k) P((1-3n+6k)^2) = 0 \tag{10}$$

is valid for all polynomials $P(x)$ of degree $n-1$. Let us consider n polynomials of degree $n-1$:

$$P_r(x) = \prod_{k=1, k\neq r}^{n} (x - (1-3n+6k)^2).$$

Using these polynomials in formula (10) we get the relations connecting the α_r with α_0. It is possible to write the answer in terms of binomial coefficients:

$$f(u) = f_0 \sum_{k=0}^{n} \binom{n-\frac{1}{3}}{k} \binom{n+\frac{1}{3}}{n-k} \sin(1-3n+6k)u \tag{11}$$

where f_0 is an arbitrary constant.

The auxiliary function $g(u) = \sin^{2n+1} u P(u)$ corresponding to the second independent solution of the T–Q equation can be found by analogy:

$$g(u) = g_0 \sum_{k=0}^{n} \binom{n-\frac{2}{3}}{k} \binom{n+\frac{2}{3}}{n-k} \sin(2-3n+6k)u. \tag{12}$$

One can easily check that $f(u)$ and $g(u)$ satisfy the ODE

$$\begin{aligned} &f'' - 6n \cot 3u \, f' + (1-9n^2) f = 0 \\ &g'' - 6n \cot 3u \, g' + (4-9n^2) g = 0. \end{aligned} \tag{13}$$

It happens that occasionally these equations are more convenient in calculations than explicit formulae (11) and (12).

The integer m in (8) is equal to the number of reversed spins and related to the z-axis projection of the total spin $S_z = N/2 - m$. The solution $Q(u)$ has degree $m = n = (N-1)/2$, consequently the corresponding eigenvector has $S_z = 1/2$. In principle we can construct this vector using QISM [9, 10]. It is also known that for the antiferromagnetic *XXZ* chain with an even N the ground state has $S_z = 0$ [11, 12]. It can be analogically conjectured that one of the two ground states for the case of N odd has $S_z = 1/2$. Note that in the interval $u \in (\pi/3, 2\pi/3)$, the weights of the six-vertex model (5) are positive. Consequently the components of the ground state vector are non-negative due to the Perron–Frobenius theorem.

In collaboration with Razumov [13], we have found explicit values for these components up to $N = 17$. They are all positive. Hence, if one of the two ground state vectors has $S_z = 1/2$ then we have identified it. Further, we conjecture that the solution $Q(u)$ we found corresponds to the ground state for $N > 17$ as well. I believe that using ODE (13) for $f(u)$ it is possible to describe the distribution of the Bethe parameters and to prove this conjecture.

For even N, the corresponding solution $Q(u)$ has degree $N/2 + 1$ and there is no Bethe vector that ensures the simple eigenvalue we have discussed.

Now we return to the main calculations. Firstly, we only know the largest eigenvalue of the transfer matrix $T(u) = \sin^N u$ for $\eta = 2\pi/3$. It is remarkable that the knowledge of the second independent solution $P(u)$ allows us to find the derivative of this eigenvalue w.r.t. η and thus the simplest correlations.

Now we consider Baxter's T–Q equation (7). It can be interpreted as a discrete version of a second-order differential equation, so we can express its coefficients in terms of two independent solutions [14, 15]:

$$\sin^N u = P(u+\eta/2)Q(u-\eta/2) - P(u-\eta/2)Q(u+\eta/2)$$
$$T(u) = P(u+\eta)Q(u-\eta) - P(u-\eta)Q(u+\eta).$$

Using these relations in a similar fashion to as in [8], we can find the T-matrix derivative w.r.t. η:

$$\begin{aligned} T'_\eta(u)|_{\eta=2\pi/3} = \tfrac{3}{2}\big\{&P(u+\pi/3)Q'(u-\pi/3) - P'(u+\pi/3)Q(u-\pi/3) \\ &+P(u-\pi/3)Q'(u+\pi/3) - P'(u-\pi/3)Q(u+\pi/3)\big\}. \end{aligned} \tag{14}$$

The derivative of the last equation w.r.t. the spectral parameter u is

$$\begin{aligned} T''_{\eta u}(u)|_{\eta=2\pi/3} = \tfrac{3}{2}\big\{&P(u+\pi/3)Q''(u-\pi/3) - P''(u+\pi/3)Q(u-\pi/3) \\ &+P(u-\pi/3)Q''(u+\pi/3) - P''(u-\pi/3)Q(u+\pi/3)\big\}. \end{aligned} \tag{15}$$

Let us use these derivatives. Averaging (6) over the ground state we obtain the energy per site that relates the correlations to the logarithmic derivative of the largest eigenvalue of the transfer matrix w.r.t. spectral parameter u:

$$E_0(\eta)/N = -2C_1^{\perp} - \cos\eta\, C_1^{\parallel} = \cos\eta - \frac{2\sin\eta}{N}\left(\frac{T'_u}{T}\right)_{u=\eta/2}. \tag{16}$$

For $\eta = 2\pi/3$ and $T(u) = \sin^N u$ this last equation is reduced to

$$E_0/N = -2C_1^{\perp}(\Delta=-1/2) + \tfrac{1}{2}\, C_1^{\parallel}(\Delta=-1/2) = -\tfrac{3}{2}. \tag{17}$$

It is difficult to find two unknowns via one equation. However, let us differentiate equation (16) w.r.t. η. H is a Hermitian operator and so one may ignore the η dependence of the eigenvector. Hence, differentiating equation (16), the matrix element of the Hamiltonian, we can ignore the η dependence of the correlations

$$E'_0(\eta) = N\sin\eta\, C_1^{\parallel} = -N\sin\eta - 2\left\{\cos\eta\left(\frac{T'_u}{T}\right) + \sin\eta\frac{\mathrm{d}}{\mathrm{d}\eta}\left(\frac{T'_u}{T}\right)\right\}_{u=\eta/2}.$$

Jimbo and Miwa [16] used this method for calculation of the correlators $C_1^{\parallel}$ and $C_1^{\perp}$ in the thermodynamic limit.

Replacing $\eta = 2\pi/3$ we obtain the last but one formula for the correlation:

$$C_1^{\parallel}(\Delta=-1/2) = -\frac{1}{3} - \frac{2}{N}\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\left(\frac{T'_u}{T}\right)_{u=\frac{\eta}{2}}\right\}_{\eta=\frac{2\pi}{3}} = 1 + \left(\frac{2}{\sqrt{3}}\right)^{N+1}\left(T'_\eta - \frac{\sqrt{3}}{N}T''_{\eta u}\right)_{u=\frac{\pi}{3},\eta=\frac{2\pi}{3}}.$$

The problem is solved in principle. Due to formulae (14) and (15), we can express the correlations in terms of the two independent solutions $Q(u)$ and $P(u)$ which are given (up to the factor $\sin^N u$) by the formulae (11) and (12). *The final answer* is

$$\langle\sigma_j^z\sigma_{j+1}^z\rangle \equiv C_1^{\|}(\Delta=-1/2) = -\frac{1}{2}+\frac{3}{2(2n+1)^2} = -\frac{1}{2}+\frac{3}{2N^2}.$$

Detailed calculations will be published elsewhere.

Taking into account that the action of the operators $\sigma_j^z\sigma_{j+1}^z$ depends upon the relative orientation of the neighbouring spins, we can easily convert the last formula into a formula for the average number of 'strings', i.e. clusters of spins with the same orientation:

$$\langle N_{\text{string}}\rangle = \frac{3}{4}\left(N-\frac{1}{N}\right).$$

Using equation (17), we get the second correlation:

$$C_1^{\perp}(\Delta=-1/2) = \langle\sigma_j^x\sigma_{j+1}^x\rangle = \langle\sigma_j^y\sigma_{j+1}^y\rangle = \frac{5}{8}+\frac{3}{8N^2}.$$

The obtained results can be generalized in different ways. Let us discuss some possibilities.

First of all let us note that all matrix elements of the Hamiltonian (3) are integers or half integers. The ground state energy is a half integer as well. It is not surprising that the normalization of the eigenvector can be chosen so that all its components are integers. This helps us to make calculations and using Mathematica we have explicitly found eigenvectors for $N \leqslant 17$. The obtained information is contained in [13]. Let us mention only one result, related to the correlations, which are called probabilities of formation of a ferromagnetic string [17]. All data are in agreement with the conjectured formula:

$$\frac{\langle a_1\, a_2\, \cdots\, a_{k-1}\rangle}{\langle a_1\, a_2\, \cdots\, a_k\rangle} = \frac{(2k-2)!\,(2k-1)!\,(2n+k)!\,(n-k)!}{(k-1)!\,(3k-2)!\,(2n-k+1)!\,(n+k-1)!}$$

where $a_j = (1+\sigma_j^z)/2$

Many remarkable connections between the wavefunction components are noticeable. As it turns out, the ratio of the largest component to the smallest one is equal to the number of ASM (*alternating sing matrices*). The wonderful history of these numbers has already interweaved with the six-vertex model (see, e.g., [18]). In all probability this is just the tip of the iceberg.

Second one can try to consider the elliptic case.

Thirdly we can consider the inhomogeneous six-vertex model.

I would like to mention that the necessity to distinguish between even and odd chains was remarked upon by Faddeev and Takhtajan [19] for the *XXX* chain, Baake *et al* [20] for *XXZ*, Bugrij [21] for the Ising model and so on. The authors of [27] derived benefit from consideration of *XXZ* spin chains with an odd number of sites. As far as more recent results are concerned I want to mention our paper [15], where it was noticed that the properties of Bethe equations depend on the parity of spin chain length. The work of Schnack *et al* [26] contains numeric investigations of the spin $1/2, 1, \ldots, 5/2$ *XXX* chain and demonstrates the peculiarities of odd N. Finally, I would like to mention the very recent paper by Albertini [28] where the author considers spin chains with open boundaries and stresses that antiferromagnetic quantum spin chains can in principle have very different properties according to the parity of their length.

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References

[1] Baxter R J 1972 *Ann. Phys., NY* **70** 323
[2] Baxter R J 1989 *Adv. Stud. Pure Math.* **19** 95
[3] Alcaraz F C, Barber M N, Batchelor M T, Baxter R J and Quispel G R W 1987 *J. Phys. A: Math. Gen.* **20** 6397
[4] Hinrichsen H, Martin P P, Rittenberg V and Scheunert M 1994 *Nucl. Phys.* B **415** 533
[5] Miguel A Martin-Delgado and German Sierra 1996 *Phys. Rev. Lett.* **76** 1146
[6] Fridkin V, Stroganov Yu G and Don Zagier 2000 *J. Phys. A: Math. Gen.* **33** L121
[7] Perk J H H and Schultz C L 1981 *Phys. Lett.* A **84** 407
[8] Fridkin V, Stroganov Yu G and Don Zagier 2001 *J. Stat. Phys.* **102**
[9] Kulish P P and Sklyanin E K 1979 *Phys. Lett.* A **70** 461
[10] Takhtajan L A and Faddeev L D 1979 *UMN* **34** 13
[11] Yang C N and Yang C P 1966 *Phys. Rev.* **150** 321
Yang C N and Yang C P 1966 *Phys. Rev.* **150** 327
[12] Lieb E H 1967 *Phys. Rev.* **162** 162
Lieb E H 1967 *Phys. Rev. Lett.* **18** 1046
Lieb E H 1967 *Phys. Rev. Lett.* **19** 108
[13] Razumov A V and Stroganov Yu G 2001 Spin chains and combinatorics *J. Phys. A: Math. Gen.* **33** at press (Razumov A V and Stroganov Yu G 2000 *Preprint* cond-mat/0012141)
[14] Krichiver I, Lipan O, Wiegmann P and Zabrodin A 1997 *Commun. Math. Phys.* **188** 267
[15] Pronko G P and Stroganov Yu G 1999 *J. Phys. A: Math. Gen.* **32** 2333
[16] Jimbo M and Miwa T 1996 *J. Phys. A: Math. Gen.* **29** 2923
[17] Korepin V E, Izergin A G and Bogoliubov N M 1993 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)
[18] Bressoud D and Propp J 1999 *Notices of the AMS* **46** 637
[19] Faddeev L D and Takhtajan L A 1981 *Phys. Lett.* A **85** 375
[20] Baake M, Christe P and Rittenberg V 1988 *Nucl. Phys.* B **300** 637
[21] Bugrij A Private communication
[22] Henkel M and Schollwock U 2000 Universal finite-size scaling amplitudes in anisotropic scaling *Preprint* cond-mat/0010061
[23] Belavin A A and Stroganov Yu G 1999 *Phys. Lett.* B **466** 281
[24] Sutherland B 1967 *Phys. Rev. Lett.* **19** 103
[25] Alcaraz F C, Barber M N and Batchelor M T 1988 *Ann. Phys., NY* **182** 280
[26] Bärwinkel K, Schmidt H-J and Schnack J 2000 *J. Magn. Magn. Mater.* **220** 227
Schnack J 2000 *Phys. Rev.* B **62**
[27] Doikou A, Mezincescu L and Nepomechie R I 1997 *J. Phys. A: Math. Gen.* **30** L507
Doikou A, Mezincescu L and Nepomechie R I 1997 *Mod. Phys. Lett.* A **12** 2591
[28] Albertini G 2000 Is the purely biquadratic spin 1 chain always gapful? *Preprint* cond-mat/0012439
[29] Feynman R P 1939 *Phys. Rev.* **56** 340
[30] Johnson J D, Krinsky S and McCoy B M 1973 *Phys. Rev.* A **8** 2526
Gaudin M, McCoy B M and Wu T T 1981 *Phys. Rev.* D **23** 417